Problem Set 1

1. Derive the OLS estimator using calculus. To do this first derive $\partial a'v/\partial v$ and $\partial v'Av/\partial v$ for any vector $a$ and any symmetric full rank matrix $A$.

2. Show for the classical multiple regression model that $\hat{\beta}$ and $(\tilde{\beta} - \hat{\beta})$ are uncorrelated with $\tilde{\beta}$ being a linear unbiased estimator.

3. Prove that the test statistic
$$
\frac{(\hat{\beta} - \beta_0)'X'X(\hat{\beta} - \beta_0)/k}{s^2} = \frac{(\hat{u}'\hat{u}_0 - \hat{u}'\hat{u})/k}{\hat{u}'\hat{u}/(n - k)}
$$
for any $\beta_0$ by looking at the geometry of the numerator.

4. Assume $u \sim N(0, \sigma^2 I)$. Derive the distribution of $R\beta$ with an $m \times k$ matrix with rank($R$) = $m < k$. Construct a scalar test statistic for testing $H_0 : R\beta = r$ and derive its distribution.

5. Consider a model with only two regressors $x_1$ and $x_2$ and the restriction $\beta_1 + \beta_2 = 4$. Show geometrically the resulting restricted and unrestricted LS estimators.

6. The correct structure of the model is assumed to be $E(y|X) = X\beta$. Since you are not sure about this you run a regression of $y$ on $X$ and $W$. What kind of effects are resulting for the estimates $\tilde{\beta}$?

7. Same setting as above. But you only use part of the regressors, i.e. you run a regression of $y$ on $X_1$ with $X = [X_1 : X_2]$. Discuss the effects on $\beta_1$.

8. Read the empirical example on individual wages in Verbeek chapter 2. Relate the two problems above to Verbeek’s empirical results.

9. Let $X$ and $W$ be two $n \times k$ matrices such that $S(X) \neq S(W)$. Show that $P \equiv X(W'X)^{-1}W'$ is idempotent but not symmetric. Characterize the spaces that $P$ and $(I - P)$ project on to.

10. Show that the estimator $\hat{\beta} = (X'BX)^{-1}X'By$ fulfills the prerequisites of the Gauss-Markov theorem. Derive the covariance matrix of $\hat{\beta}$. 
Problem Set 2

1. Let $Z_i \sim B(1; 0.8)$ and $\bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i$. Compute $E(\bar{Z})$ and $\text{Var}(\bar{Z})$, and the analogue for $\bar{W}$ where $W_i = \exp(Z_i)$ for $n = 1$ and $n = 3$. If you have e.g. EXCEL available perform these calculations also for $n = 10$, $n = 100$, $n = 250$, and $n = 500$, respectively.

   - Relate those results to Jensen’s inequality.
   - Explain the asymptotic behavior of the random variable $\exp(\bar{Z})$.
   - How do we see the $\sqrt{n}$ convergence rate?
   - Visualize the probability distribution of $\bar{Z}$.
   - Explain the asymptotic behavior of the random variable $\sqrt{n} \exp(\bar{Z})$.

2. Give answers to Verbeek’s Exercise 5.3.
Problem Set 3

1. In a classical linear regression setup a second (artificial) regression is performed as

\[ y - X\hat{\beta}_{OLS} = Xb + \text{Residuals} \]

What is the resulting OLS estimator \( \hat{b} \), the sum of squared residuals, and the estimated standard errors of \( \hat{b} \)?

2. How should such an artificial regression look like for a non-linear regression model

\[ y = x(\beta) + u \]

where \( u \) has the classical properties?

3. In a linear regression model with autoregressive residuals of order \( p \)

\[ u_t = \rho_1 u_{t-1} + \ldots + \rho_p u_{t-p} + \varepsilon_t \quad \varepsilon_t \sim IID(0, \sigma^2_\varepsilon) \]

firstly argue that the model can be interpreted as a non-linear regression model. Secondly, derive an LM-type test for \( H_0 : \rho_1 = \ldots = \rho_p = 0 \).

4. In a linear regression model with heteroscedasticity of the form

\[ \sigma_t = \sigma^2 h(Z_t\gamma) \quad \text{with } h(0) = 1 \]

derive the score vector and also the information matrix \( I(\theta) \) with \( \theta = (\beta', \gamma', \sigma^2)' \).

Show that the LM test statistic is equivalent to \( nR^2 \) where \( R^2 \) is the uncentred \( R^2 \) of an auxiliary regression of a vector of ones on the score vectors \( G \) where the \( t \)-th row of \( G \) is the individual score contribution \( g_t(\theta)' \). [Hint: See section 6.2.2. in V].

5. Describe how you would compute the feasible GLS estimator for \( \beta \) for the above model with autoregressive residuals. (Do not worry about the initial observations.)